HW 5 SOL'S

SECTION 2.2

(10) $y'' + 2y' + 2y = 0$

$\rightarrow r^2 + 2r + 2 = 0$

$\Rightarrow r = \dfrac{-2 \pm \sqrt{4 - (4)(1)(2)}}{2} = -1 \pm i$

$\therefore \boxed{y(t) = [c_1 \cos(t) + c_2 \sin(t)]e^{-t}}$

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(4) $y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2.$

$\rightarrow r^2 - 2r + 5 = 0 \quad \Rightarrow \quad r = \dfrac{2 \pm \sqrt{4 - (4)(1)(5)}}{2} = 1 \pm 2i$

$\Rightarrow y(t) = [c_1 \cos(2t) + c_2 \sin(2t)]e^{t}$

$y(\pi/2) = 0 = (c_1 \cos(\pi) + c_2 \sin(\pi))e^{\pi/2} = e^{\pi/2} \cdot (-c_1 + 0)$

$\Rightarrow \boxed{c_1 = 0.}$

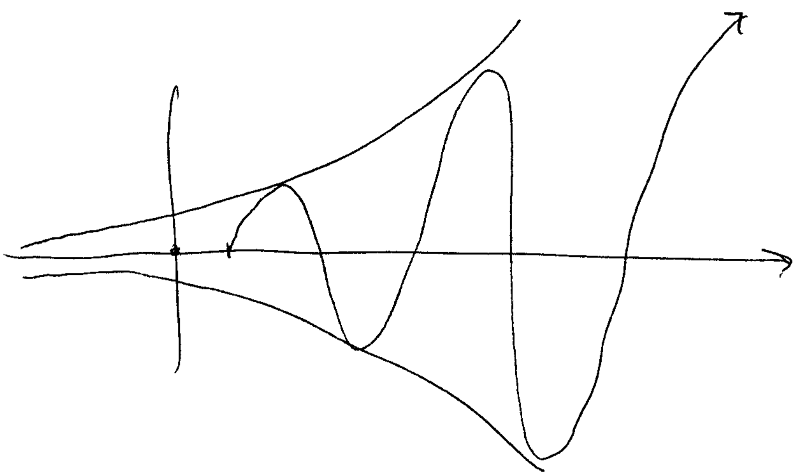
$\Rightarrow y(t) = e^{t} \cdot c_2 \cdot \sin(2t) \Rightarrow y'(t) = c_2 e^{t} \sin(2t) + c_2 e^{t}(2\cos(2t))$

(4) (CONTINUED)

$$y'(\pi/2) = 2 = c_2 e^{\pi/2} \cancel{\sin(\pi)} + c_2 e^{\pi/2} \cdot 2 \cdot \cos(\pi)$$

$$\Rightarrow \quad 2 = -2 c_2 e^{\pi/2} \quad \Rightarrow \quad \boxed{c_2 = -e^{-\pi/2}}$$

$$\therefore \quad \boxed{y(t) = -e^{t} \cdot e^{\pi/2} \cdot \sin(2t)}$$



SOL^{n} OSCILLATES WITH LARGER AND LARGER AMPLITUDE AS $t \to \infty$

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(6) $$W\left[e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)\right] = \begin{vmatrix} e^{\lambda t}\cos(\mu t) & e^{\lambda t}\sin(\mu t) \\ \lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin(\mu t) & \lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t) \end{vmatrix}$$

$$= \lambda e^{2\lambda t}\cos\mu t \cdot \sin\mu t + \mu e^{2\lambda t}\cos^2(\mu t) - \left[\lambda e^{2\lambda t}\cos(\mu t)\sin(\mu t) - \mu e^{2\lambda t}\sin^2(\mu t)\right]$$

$$= 0 + \mu e^{2\lambda t}\left(\cos^2(\mu t) + \sin^2(\mu t)\right) = \boxed{\mu e^{2\lambda t}}$$

SECTION 2.2

(8) $t^2 y'' + 3t y' + y = 0 \quad (*)$, $t > 0$, $y_1(t) = t^{-1}$

WE KNOW y_1 SOLVES $(*)$. LET $y_2(t) = v(t)\, y_1(t)$

THEN $y_2' = v' y_1 + v y_1'$

$$y_2'' = v'' y_1 + 2v' y_1' + v y_1''$$

PLUG y_2 INTO $(*)$ TO GET:

$$t^2 \left(v'' y_1 + 2 v' y_1' + v y_1'' \right) + 3t \left(v' y_1 + v y_1' \right) + v y_1 = 0.$$

$\Rightarrow$

$$\underbrace{\left[t^2 y_1'' + 3t y_1' + y_1 \right]}_{=0 \text{ B.C. } y_1 \text{ SOLVES } (*).} v + t^2 v'' y_1 + 2t^2 v' y_1' + 3t v' y_1 = 0$$

NOW PLUG IN $y_1 = t^{-1}$
$y_1' = -t^{-2}$.

$t v'' + (-2v') + 3v' = t v'' + v' = 0$. LET $w = v'$

THEN $t w' + w = 0$.

8 (CONTINUED)

$$t w' = -w \implies \int \frac{dw}{w} = \int -\frac{1}{t}\, dt$$

$$\implies \ln(w) = -\ln t = \ln(t^{-1})$$

$$\implies w = t^{-1} \qquad \text{RECALL } w = v'$$

$$\implies v' = t^{-1} \implies v(t) = \ln(t).$$

$$\therefore y_2 = v(t)\, y_1(t) = \ln(t) \cdot t^{-1}$$

$$\boxed{y_2 = \frac{\ln(t)}{t}}$$

SECTION 2.5

4 $u'' + \omega_0^2 u = \cos(\omega t)$, $\omega^2 \neq \omega_0^2$.

HOMOGENEOUS EQ$^{\underline{n}}$: $u_H'' + \omega_0^2 u_H = 0. \implies r^2 + \omega_0^2 = 0$

$$\implies r = \pm \omega_0 i$$

SO $\boxed{u_H = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)}$

NOW WE NEED PARTICULAR SOL$^{\underline{n}}$.

4 (CONTINUED)

SINCE $\omega_0^2 \neq \omega^2$, WE GUESS $u_p = A\cos(\omega t) + B\sin(\omega t)$

$$\Rightarrow u_p'' = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$$

PLUG INTO EQ$^{\text{N}}$:

$$-\omega^2\left(A\cos(\omega t) + B\sin(\omega t)\right) + \omega_0^2\left(A\cos(\omega t) + B\sin(\omega t)\right) = \cos(\omega t)$$

$$\Rightarrow A(\omega_0^2 - \omega^2)\cos(\omega t) + B(\omega_0^2 - \omega^2)\sin(\omega t) = \cos(\omega t)$$

$$\Rightarrow \begin{array}{l} A(\omega_0^2 - \omega^2) = 1 \\ B(\omega_0^2 - \omega^2) = 0 \end{array} \Rightarrow \begin{array}{l} A = \dfrac{1}{\omega_0^2 - \omega^2} \\ B = 0 \end{array}$$

$$\Rightarrow u_p = \frac{1}{\omega_0^2 - \omega^2}\cos(\omega t)$$

$$\therefore \boxed{u(t) = u_H + u_p = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \left(\frac{1}{\omega_0^2 - \omega^2}\right)\cos(\omega t)}$$

(6) $y'' + y' - 2y = 2t \,, \quad y(0) = 0, \quad y'(0) = 1$

HOM. SOL^{n}:

$$y_H = c_1 e^{t} + c_2 e^{-2t}$$

LET $y_p = At + B$

$y_p' = A$

$y_p'' = 0$

$$\Rightarrow \quad 0 + A - 2(At + B) = 2t$$

$$-2At + A - 2B = 2t$$

$$\Rightarrow \quad -2A = 2 \quad \Rightarrow \quad \boxed{A = -1}$$

$$A - 2B = 0 \qquad \boxed{B = -1/2}$$

$$\therefore \quad y(t) = c_1 e^{t} + c_2 e^{-2t} - t - 1/2$$

$$y(0) = c_1 + c_2 - 1/2 = 0$$

$$y'(0) = c_1 - 2c_2 - 1 = 1$$

$$\Rightarrow \boxed{\begin{aligned} c_1 &= 1 \\ c_2 &= -1/2 \end{aligned}}$$

$$\Rightarrow \boxed{y(t) = e^{t} - 1/2 e^{-2t} - t - 1/2}$$

(11) $\underbrace{y'' - 3y' - 4y} = 3e^{2t}$

$$= (D^2 - 3D - 4)y = (D-4)\underbrace{(D+1)y}_{=u} = 3e^{2t}$$

$$(D-4)u = 3e^{2t} \Rightarrow u' - 4u = 3e^{2t}$$

$$\Rightarrow u \cdot e^{-4t} = \int 3e^{-2t} = -\frac{3}{2}e^{-2t}$$

$$\Rightarrow u = -\frac{3}{2}e^{2t}$$

THEN $(D+1)y = u \Rightarrow y' + y = -3/2\, e^{2t}$

$$\Rightarrow ye^{t} = -\frac{3}{2}\int e^{3t}\,dt = -1/2\, e^{3t}$$

$$\Rightarrow \boxed{y = -\frac{1}{2}e^{2t}}$$

NOW COMBINE THIS PARTICULAR SOL^{N} W/ HOM. SOL^{NS} TO GET:

$$\boxed{y(t) = c_1 e^{4t} + c_2 e^{-t} + (-1/2\, e^{2t})}$$

M427J: Differential Equations with Linear Algebra
Homework # 05
Handout: 02/14/2017, Tuesday
Due: 03/01/2017, Wednesday

• **Submission:** Please make your homework neat and **STAPLED**. You have to submit your homework **Wednesday** in the Problem Session. Note that *no late homework will be accepted.*

• **Assignments for Section 2.2: The Homogeneous Equations with Constant Coefficients (II)**

In each of the following problems find the general solution of the given differential equation.

1. $y'' - 2y' + 6y = 0$
2. $y'' + 6y' + 13y = 0$
3. $y'' + y' + 1.25y = 0$

10. $y'' + 2y' + 2y = 0$
12. $4y'' + 9y = 0$
16. $y'' + 4y' + 6.25y = 0$

In each of the following problems find the solution of the given differential equation. Sketch the graph of the solution and describe its behavior for increasing t.

4. $y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$
5. $y'' + 2y' + 2y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$

6. Show that $W(e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)) = \mu e^{2\lambda t}$.

7. Using Euler's formula, show that

$$\cos t = (e^{it} + e^{-it})/2, \qquad \sin t = (e^{it} - e^{-it})/(2i).$$

• **Assignments for Section 2.2: The Homogeneous Equations with Constant Coefficients (III)**

In each of the following problem, find the general solution of the given differential equation.

1. $y'' - 2y' + y = 0$
2. $4y'' - 4y' - 3y = 0$
3. $4y'' + 12y' + 9y = 0$
4. $16y'' + 24y' + 9y = 0$
5. $2y'' + 2y' + y = 0$

6. Consider the initial value problem

$$9y'' + 12y' + 4y = 0, \qquad y(0) = a > 0, \qquad y'(0) = -1.$$